Supplement to the paper "Floating bundles and their applications"

A. V. Ershov

This paper is the supplement to the section 2 of the paper "Floating bundles and their applications" [2]. Below we construct the denumerable set of extensions of the formal group of geometric cobordisms $F(x\otimes 1, 1\otimes x)$ by the Hopf algebra $H=\Omega_U^*(Gr)$.

Let $F_i(x\otimes 1, 1\otimes x),\ i=1,2$ be formal groups over ring R. Recall the following definition.

Definition 1. A homomorphism of formal groups $\varphi\colon F_1\rightarrow F_2$ is a formal series $\varphi(x)\in R[[x]]$ such that $\varphi(F_1(x\otimes 1, 1\otimes x))=F_2(\varphi(x)\otimes 1, 1\otimes \varphi(x))$.

Let H be a Hopf algebra over ring R with diagonal Δ; let $\mathfrak{F}_i(x\otimes 1, 1\otimes x),\ i=1,2$ be formal groups over H.

Definition 2. A homomorphism of formal groups over Hopf algebra H $\Phi\colon \mathfrak{F}_1\rightarrow \mathfrak{F}_2$ is a formal series $\Phi(x)\in H[[x]]$ such that $(\Delta\Phi)(\mathfrak{F}_1(x\otimes 1, 1\otimes x))=\mathfrak{F}_2(\Phi(x)\otimes 1, 1\otimes \Phi(x))$.

Note that $\varepsilon(\Phi)\colon (\varepsilon\otimes\varepsilon)(\mathfrak{F}_1)\rightarrow(\varepsilon\otimes\varepsilon)(\mathfrak{F}_2)$ is the homomorphism of the formal groups over the ring R (where ε is the counit of the Hopf algebra H). We say that the homomorphism Φ covers the homomorphism $\varepsilon(\Phi)$.

Let R be the ring $\Omega_U^*(\mathrm{pt})$; let $F(x\otimes 1, 1\otimes x)\in R[[x\otimes 1, 1\otimes x]]$ be the formal group of geometric cobordisms. Let H be the Hopf algebra $\Omega_U^*(Gr)$. By definition, put $\varphi^{(1)}(x)=x,\ \varphi^{(-1)}(x)=\theta(x)$ and $\varphi^{(n)}(x)=F(x,\varphi^{(n-1)}(x))$, where $\theta(x)\in R[[x]]$ is the inverse element in F. Clearly, that $\varphi^{(n)}\colon F\rightarrow F$ is

the homomorphism for every $n \in \mathbb{Z}$. Power systems were considered by S. P. Novikov and V. M. Buchstaber in [1].

Below for any $n \in \mathbb{Z}$ we construct the extension $\mathfrak{F}^{(n)}(x \otimes 1, 1 \otimes x)$ of $F(x \otimes 1, 1 \otimes x)$ by H and the homomorphism $\Phi^{(n)} \colon \mathfrak{F} \to \mathfrak{F}^{(n)}$ such that

(i) $\mathfrak{F}^{(1)} = \mathfrak{F}$;

(ii) $\varepsilon(\Phi^{(n)}) = \varphi^{(n)}$.

Let X be a finite CW-complex. Recall that the set of FBSP over X is the semigroup with respect to the multiplication of FBSP [2]. Let n be a positive integer. Let us take the product of the FBSP $\widetilde{Gr}_{k,kl}$ (over $Gr_{k,kl}$) with itself n times. It is the FBSP over $Gr_{k,kl}$ with a fiber $\mathbb{C}P^{k^n-1} \times \mathbb{C}P^{l^n-1}$. By $\widetilde{Gr}^{(n)}_{k,kl}$ denote the obtained FBSP. Let $\widehat{Gr}^{(n)}_{k,kl}$ be the corresponding bundle over $Gr_{k,kl}$ with fiber $\mathbb{C}P^{k^n-1}$. Let $\widehat{Gr}^{(n)} = \varinjlim_{(k,l)=1} \widehat{Gr}^{(n)}_{k,kl}$. We have the evident fiber maps $\widehat{Gr}_{k,kl} \to \widehat{Gr}^{(n)}_{k,kl}$, $\lambda^{(n)} \colon \widehat{Gr} \to \widehat{Gr}^{(n)}$ and the following commutative diagrams ($(km, ln) = 1$):

$$\begin{array}{ccc} \widehat{Gr}_{km,klmn} & \to & \widehat{Gr}^{(n)}_{km,klmn} \\ \uparrow & & \uparrow \\ \widehat{Gr}_{k,kl} \times \widehat{Gr}_{m,mn} & \to & \widehat{Gr}^{(n)}_{k,kl} \times \widehat{Gr}^{(n)}_{m,mn}, \end{array} \tag{1}$$

$$\begin{array}{ccc} \widehat{Gr} & \stackrel{\lambda^{(n)}}{\to} & \widehat{Gr}^{(n)} \\ {\scriptstyle \widehat{\phi}}\uparrow & & \uparrow{\scriptstyle \widehat{\phi}^{(n)}} \\ \widehat{Gr} \times \widehat{Gr} & \stackrel{\lambda^{(n)} \times \lambda^{(n)}}{\to} & \widehat{Gr}^{(n)} \times \widehat{Gr}^{(n)}. \end{array} \tag{2}$$

By x denote the class of cobordisms in $\Omega^2_U(\widehat{Gr}^{(n)})$ such that its restriction to any fiber $\cong \mathbb{C}P^\infty$ is the generator $x \mid_{\mathbb{C}P^\infty} \in \Omega^2_U(\mathbb{C}P^\infty)$. Let $\Phi^{(n)}(x) \in H[[x]]$ be the series, defined by the fiber map $\lambda^{(n)}$ (see [2]). Let

$$\mathfrak{F}^{(n)}(x \otimes 1, 1 \otimes x) \in H \widehat{\otimes}_R H[[x \otimes 1, 1 \otimes x]]$$

be the series, corresponds to the fiber map $\widehat{Gr}^{(n)} \times \widehat{Gr}^{(n)} \stackrel{\widehat{\phi}^{(n)}}{\to} \widehat{Gr}^{(n)}$ (see [2]; note that $Gr^{(n)}$ is the H-group with the multiplication $\widehat{\phi}^{(n)}$). Clearly, that

$\mathfrak{F}^{(n)}(x\otimes 1, 1\otimes x)$ is an extension of $F(x\otimes 1, 1\otimes x)$ by H (in particular, it is the formal group over Hopf algebra H). Note that $\lambda^{(n)}$ covers the identity map of the base Gr. It follows from diagram (2) that

$$(\Delta\Phi^{(n)})(\mathfrak{F}(x\otimes 1, 1\otimes x)) = \mathfrak{F}^{(n)}(\Phi^{(n)}(x)\otimes 1, 1\otimes\Phi^{(n)}(x)).$$

It is clear that $\varepsilon(\Phi^{(n)})(x) = \varphi^{(n)}(x)$.

For $n = 0$ let $\widehat{Gr}^{(0)} = Gr\times\mathbb{C}P^\infty$ and let $\lambda^{(0)}$ be the composition

$$\widehat{Gr}\to \mathrm{pt}\to\widehat{Gr}^{(0)}.$$

It defines the series $\mathfrak{F}^{(0)} = F$ and $\Phi^{(0)} = 0$.

Let $\lambda^{(-1)}$ be the fiber map $\widehat{Gr}\to\widehat{Gr}^{(-1)} = \widehat{Gr}$ such that the following conditions hold:

(i) the restriction of $\lambda^{(-1)}$ to any fiber is the inversion in the H-group $\mathbb{C}P^\infty$ (i. e. the complex conjugation);

(ii) $\lambda^{(-1)}$ covers the map $\nu\colon Gr\to Gr$, where ν is the inversion in the H-group Gr.

Let $\Phi^{(-1)}(x)\in H[[x]]$ be the series, defined by $\lambda^{(-1)}$. Trivially, that $\varepsilon(\Phi^{(-1)})(x) = \theta(x)$. Note that the $\lambda^{(-1)}$ coincides with $\widehat{\nu}$ (see [2]). Consequently, $\Phi^{(-1)} = \Theta(x)$. Now we can define $\mathfrak{F}^{(n)}$ and $\Phi^{(n)}$ for negative integer n by the obvious way.

By S denote the antipode of the Hopf algebra H. Let μ be the multiplication in the Hopf algebra H. By definition, put $(1) = \mathrm{id}_H$, $(-1) = S\colon H\to H$ and $(n) = \mu\circ((n-1)\otimes(1))\circ\Delta\colon H\to H$ (in particular, $(0) = \eta\circ\varepsilon\colon H\to H$, where η is the unit in H).

Proposition 3. *$\mathfrak{F}^{(n)}(x\otimes 1, 1\otimes x) = (((n)\otimes(n))\mathfrak{F})(x\otimes 1, 1\otimes x)$ for any $n\in\mathbb{Z}$.*

Proof. By $\phi\colon Gr\times Gr\to Gr$ denote the multiplication in the H-space Gr. Suppose n a positive integer. By definition, put $\phi(1) = \mathrm{id}_{Gr}$, $\phi(n) = \phi\circ(\phi(n-1)\times\mathrm{id}_{Gr})$, and $\mathrm{diag}\,(n) = (\mathrm{diag}\,(n-1)\times\mathrm{id}_{Gr})\circ\mathrm{diag}$, where $\mathrm{diag}(1) = \mathrm{id}_{Gr}$, $\mathrm{diag} = \mathrm{diag}\,(2)\colon Gr\to Gr\times Gr$. Note that the composition $\phi(n)\circ\mathrm{diag}\,(n)\colon Gr\to Gr$ induces the homomorphism $(n)\colon H\to H$.

Let us consider the classifying map $\alpha(n)\colon Gr \to Gr$ for the bundle $\widehat{Gr}^{(n)}$ over Gr. We have the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{C}P^{\infty} \arrow[r, "="] \arrow[dr] & \mathbb{C}P^{\infty} \arrow[dr] & \\
& \widehat{Gr}^{(n)} \arrow[r, "\widehat{\alpha}(n)"] \arrow[d] & \widehat{Gr} \arrow[d] \\
& Gr \arrow[r, "\alpha(n)"] & Gr
\end{tikzcd}
$$

It is easy to prove that $\alpha(n) = \phi(n) \circ \operatorname{diag}(n)$. Hence $\alpha(n)^* = (n)\colon H \to H$. Note that the following diagram

$$
\begin{array}{ccc}
\widehat{Gr}^{(n)} \times \widehat{Gr}^{(n)} & \overset{\widehat{\alpha}(n)\times\widehat{\alpha}(n)}{\to} & \widehat{Gr} \times \widehat{Gr} \\
{\scriptstyle \widehat{\phi}^{(n)}}\downarrow & & \downarrow{\scriptstyle \widehat{\phi}} \\
\widehat{Gr}^{(n)} & \overset{\widehat{\alpha}(n)}{\to} & \widehat{Gr}
\end{array} \tag{3}
$$

is commutative. This completes the proof for positive n. For negative n proof is similar. $\square$

We can define the structure of group on the set $\{\mathfrak{F}^{(n)};\ \ n \in \mathbb{Z}\}$ in the following way. Recall that for any Hopf algebra H the triple $(\mathrm{Hom}_{Alg.Hopf}(H,H), \star, \eta \circ \varepsilon)$ is the algebra with respect to the convolution $f \star g = \mu \circ (f \otimes g) \circ \Delta\colon H \to H$. It follows from the previous Proposition that the formal group $\mathfrak{F}^{(n)}$ corresponds to the homomorphism $(n)\colon H \to H$ (see Conjecture 24 in [2]). Clearly, that $(m) \star (n) = (m+n)$ for any $m, n \in \mathbb{Z}$.

References

[1] V. M. Buchstaber, S. P. Novikov Formal groups, power systems and operators of Adams.— Matematichesky sbornik (novaia seria) T. 84(126):1, 1971. (in Russian)

[2] A. V. Ershov Floating bundles and their applications.— arXiv:math.AT/0102054